OKOUNKOV BODIES FOR AMPLE LINE BUNDLES

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ABSTRACT. Let $\mathscr{L} \to X$ be an ample line bundle over a nonsingular complex projective variety X. We construct an admissable flag $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$ of subvarieties for which the associated Okounkov body for $\mathscr{L}$ is a rational polytope.

1. INTRODUCTION

The *Okounkov body* of a line bundle $\mathscr{L} \to X$ over a nonsingular n-dimensional complex variety is a convex set in $\mathbb{R}^n$ which carries information about the section ring $\bigoplus_{k=0}^{\infty} H^0(X, \mathscr{L}^k)$ of $\mathscr{L}$. The idea is to construct a valuation-like function (meaning that it has the properties of a valuation in the ring-theoretic sense, even if it only defined on homogeneous elements)

$$v : \bigsqcup_{k \in \mathbb{N}} H^0(X, \mathscr{L}^k) \setminus \{0\} \to \mathbb{N}_0^n,$$

and define the semigroup

$$S(\mathscr{L}, v) := \{(k, v(s)) \mid s \in H^0(X, \mathscr{L}^k) \setminus \{0\}\}.$$

The associated Okounkov body is then defined as

$$\Delta(\mathscr{L}, v) := \overline{\mathrm{conv}} \left\{ \frac{1}{k}(k, v(s)) \mid s \in H^0(X, \mathscr{L}^k) \setminus \{0\} \right\}. \tag{1}$$

Roughly speaking, the function v will be defined as the successive orders of vanishing of sections along a flag $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-1} \subseteq X_n = X$ of irreducible nonsingular subvarieties such that X_i has dimension i.

These bodies were introduced by Okounkov in [Ok96] from a representation-theoretic point of view. However, he only considered the semigroup defined by the values of U-invariants, where U is the unipotent radical of a Borel subgroup, B, of G. It turned out that the elements of height k in this semigroup carry information about the decomposition of the kth piece, and that the Euclidean volumes certain slices of the associated convex body describe the asymptotics of the decomposition.

It has since then become an interesting problem *per se* to study the semigroups and associated convex bodies defined by considering the values of all sections of a line bundle, without the assumption of a group action ([LM09]), and their relation to the geometry of the line bundle $\mathscr{L}$. One crucial connection between the section ring of $\mathscr{L}$ and the Okounkov body $\Delta(\mathscr{L}, v)$ is

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the identity

$$\mathrm{vol}\Delta(\mathscr{L},v)=\lim_{k\to\infty}\frac{\dim H^0(X,\mathscr{L}^k)}{k^n}, \tag{2}$$

which holds for the linear series of a big line bundle over a smooth irreducible projective variety X (see [LM09]).

A fundamental question to ask is whether $\Delta(\mathscr{L},v)$ is a rational polytope, i.e., the convex hull of finitely many points in $\mathbb{Q}^n$. This was conjectured by Okounkov in [Ok96] in his setting where he only considered U-invariants. It is true in some cases, e.g., equivariant line bundles over toric varieties. This follows from recent results by Kaveh and Khovanskii where they consider a setting that generalizes Okounkov's, namely group representations on graded algebras of meromorphic functions on X, cf. [KK10]. In fact, the Okounkov body of a torus-equivariant line bundle over a projective toric variety equals the *moment polytope* (cf. [Br86]) associated to the graded representation. Recently Kaveh ([Kav11]) has studied this problem for a line bundle over a full flag variety G/B. He considers a flag of translated Schubert varieties and shows, using Bott-Samelson resolutions, that the corresponding Okounkov body is a polytope by identifying it with a Littelmann string polyope.

It is however not true in general that $\Delta(L,v)$ is a rational polytope. A counterexample can be found in [LM09, Section 6.3].

In this short note, we prove that if $\mathscr{L}$ is ample, there exists a flag of nonsingular irreducible subvarieties $X_0\subseteq X_1\subseteq\cdots\subseteq X_n:=X$ for which the associated Okounkov body is a rational polytope.

We would like to point out that the result in this note also follows from a more general result obtained by Anderson, Küronya and Lozovanu in [AKL12], where they study the problem of finding a finitely generated semigroup defining the Okounkov body.

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2. A polyhedral Okounkov body

Let X be a nonsingular n-dimensional complex variety and let $\mathscr{L}\to X$ be an ample line bundle. Since Okounkov bodies satisfy the property $\Delta(\mathscr{L}^k)=k\Delta(\mathscr{L})$, $k\in\mathbb{N}$ (cf. [LM09, Proposition 4.1.]), we may without loss of generality assume that $\mathscr{L}$ is very ample.

By Bertini's theorem there exist sections $\xi_1,\ldots,\xi_{n-1}\in H^0(X,\mathscr{L})$ such that the zero sets

$$X_i:=\{x\in X\mid \xi_1(x)=\cdots=\xi_{n-i}(x)=0\},\quad i=1,\ldots,n-1 \tag{3}$$

define nonsingular, irreducible subvarieties of X. Let $X_0:=\{p\}\subseteq X_1$ for some point $p\in X_1$. Let

$$v:\bigsqcup_{k\geq 0}H^0(X,\mathscr{L}^k)\setminus\{0\}\to\mathbb{N}^n$$

be the valuation-like function defined by the flag

$$X_0\subseteq X_1\subseteq\cdots\subseteq X_{n-1}\subseteq X_n:=X, \tag{4}$$

and let $\Delta \subseteq \mathbb{R}^n$ be the associated Okounkov body. For the construction, we refer to [LM09]. Similarly, let

$$v_1 : \bigsqcup_{k\geq 0} H^0(X_1, \mathscr{L}^k|_{X_1}) \setminus \{0\} \to \mathbb{N}$$

be the valuation-like function defined by the flag $X_0 \subseteq X_1$, and let $\Delta_1 \subseteq \mathbb{R}$ be the corresponding Okounkov body. Then

$$\Delta_1 = [0, b] \subseteq \mathbb{R},$$

where $b = \deg \mathscr{L}|_{X_1}$ (cf. [LM09, Ex. 1.14]).

Lemma 2.1. *The inclusion*

$$[0, b] \times \{0\} \times \cdots \times \{0\} \subseteq \Delta$$

holds.

Proof. By the closedness of Δ it suffices to prove that $(c, 0, \ldots, 0) \in \Delta$ for every $0 < c < b$.

By the ampleness of $\mathscr{L}$ there exists an $N_0 \in \mathbb{N}$ such that the restriction maps

$$R_j : H^0(X, \mathscr{L}^j) \to H^0(X_1, \mathscr{L}^j|_{X_1})$$

are surjective for $j \geq N_0$. If now $c \in (0, b)$, choose a point $v_1(\tau)/m \in (c, b)$ for a section $\tau \in H^0(X_1, \mathscr{L}^m|_{X_1})$. Hence, for sufficiently large $N \in \mathbb{N}$, there exists a section $\widetilde{\tau_N} \in H^0(X, \mathscr{L}^{Nm})$ with $R_{Nm}(\widetilde{\tau_N}) = \tau^N$. Then $v(\widetilde{\tau_N}) = (v_1(\tau^N), 0, \ldots, 0) = (Nv_1(\tau), 0, \ldots, 0)$, so that $(v_1(\tau)/m, 0, \ldots, 0) \in \Delta$. By the convexity of Δ we then have $(c, 0, \ldots, 0) \in \Delta$. $\square$

Remark 2.2. The lemma would of course also hold if the subvarieties X_i were defined by unequal line bundles $\mathscr{L}_i$, $i = 1, \ldots, n-1$. For a related result for line bundles $\mathscr{L}$ that are not ample, but merely big, cf. [J10, Theorem B].

Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$.

Theorem 2.3. *The Okounkov body Δ is the convex hull of the set*

$$\{0, be_1, e_2, \ldots, e_n\}.$$

Proof. First of all, $v(\xi_{n-i+1}) = e_i$ for $i = 2, \ldots, n$. Since the linear system defined by $\mathscr{L}$ is basepoint-free, $0 \in v(H^0(X, \mathscr{L}) \setminus \{0\})$. By Lemma 2.1 $be_1 \in \Delta$. Hence the convex hull of the points $0, be_1, e_2, \ldots, e_n$ is a subset of Δ.

It thus suffices to prove that for every $a = v(s)$ for some $s \in H^0(X, \mathscr{L}^k) \setminus \{0\}$ there exist $x_0, \ldots, x_n \geq 0$ such that

$$\begin{aligned} &x_0 + \cdots + x_n = k, \\ &a = x_0 \cdot 0 + x_1 be_1 + x_2 e_2 + \cdots + x_n e_n. \end{aligned} \tag{5}$$

For this, let $a = (a_1, \ldots, a_m, 0, \ldots, 0) = v(s)$ for $s \in H^0(X, \mathscr{L}^k)$ and assume that $a_m \geq 1$. Then $(s/\xi_{n-m+1}^{a_m})|_{X_{m-1}}$ defines a section in $H^0(X_{m-1}, \mathscr{L}^{k-a_m}|_{X_{m-1}})$, and by iteration we can write

$$a = (a_1, 0, \ldots, 0) + \sum_{i=2}^{m} a_i v(\xi_{n-i+1}),$$

where $a_1 = v_1(t)$ for the section

$$t = (s/(\xi_{n-1}^{a_2} \cdots \xi_{n-m+1}^{a_m})) \mid_{X_1} \in H^0(X_1, \mathscr{L}^{k-\sum_{i=2}^m a_i} \mid_{X_1}).$$

Since the line bundle $\mathscr{L}^{k-\sum_{i=2}^m a_i} \mid_{X_1}$ is effective we must have $p := k - \sum_{i=2}^m a_i \geq 0$. Indeed, $\mathscr{L}^r \mid_{X_1}$ is effective for every $r \geq 0$, and hence $\mathscr{L}^r \mid_{X_1}$ cannot be effective for any $r < 0$. Since $\Delta_1 = [0, b]$ there exist real $x_0, x_1 \geq 0$ with $x_0 + x_1 = p$ and

$$a_1 = x_0 \cdot 0 + x_1 b., \tag{6}$$

and hence

$$a_1 e_1 = x_0 \cdot 0 + x_1 b e_1.$$

By putting

$$x_i := a_i \quad i = 2, \ldots, n,$$

we have thus found nonnegative real numbers $x_0, \ldots, x_n$ satisfying (5). □

3. Concluding remarks

The subvarieties X_i occurring in the flag (4) are unfortunately only defined very implicitly. It would be interesting to find more explicit examples of admissable flags that lead to polyhedral Okounkov bodies. In particular, in the original setting developed by Okounkov, involving the presence of a group action, it is desirable to have an admissable flag with a certain invariance property in order to relate the Okounkov body to asymptotics of multiplicities for irreducible representations.

Another interesting problem is to find admissable flags yielding polyhedral Okounkov bodies for a larger class of line bundles, or even polyhedral global Okounkov bodies (cf. [LM09, Section 4.2]).

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